Symbolic Logic

Appendix E
Outline

• Propositional logic
  • Mathematical induction

• Predicate logic

• Temporal logic systems
  • CTL
Propositional Logic

• *Proposition* is an atomic, declarative sentence that can be shown to be true or false but not both
  • “There was not a cloud in the sky today”
• Represent as $p$ or $q$, usually with subscripts
• Connectives:
  • $\neg$, or *negation* (not) [highest precedence]
  • $\lor$, or *disjunction* (and) [this and conjunction have the same precedence]
  • $\land$, or *conjunction* (or) [this and disjunction have the same precedence]
  • $\rightarrow$, or *implication* (if ... then ...) [lowest precedence]
  • (, ) group operands and operators in the usual way
Terms

- **Natural deduction**, a means of reasoning about propositions
- **Proof rules**, rules letting infer formulas from other formulas
- **Premises**, formulas we know or assume to be true to reach a conclusion (formula) we want to establish
- **Contradiction**, a formula that is always false; denoted by \( \bot \) (**bottom**)
- **Tautology**, a formula that is always true; denoted by \( T \) (**top**)

Examples

• $p \land \neg p = \bot$
  • A contradiction, $p$ and $\neg p$ cannot both be true

• $p \lor \neg p = T$
  • A tautology, as either $p$ or $\neg p$ will be true
Rules of Natural Deduction

1. If $p$ and $q$ are true, so is $p \land q$ (conjunction introduction rule)
2. If $p \land q$ is true, so is $p$ and so is $q$ (conjunction elimination rule)
3. If $p$ is true, so is $p \lor q$; if $q$ is true, so is $p \land q$ (disjunction introduction rule)
4. If $p \lor q$ is true, and we want to conclude $Q$, we assume $p$ and conclude $Q$; then we assume $q$ and conclude $Q$. Given $p \lor q$ and these two proofs, we can infer $Q$ (disjunction elimination rule)
Rules of Natural Deduction

5. Assume $p$ is true temporarily and based on this assumption prove $q$. Then we can conclude $p \rightarrow q$ (implication introduction)

6. If we can conclude $p$ and $p \rightarrow q$, then we can conclude $q$. (modus ponens; also implication elimination)

7. If we assume $p$ and conclude $\bot$, then we infer $\neg p$ (negation introduction)

8. If we assume $p$ and $\neg p$, then we conclude $\bot$ (negation elimination)
Rules of Natural Deduction

9. If we assume \( \bot \), then we can prove any \( p \). (*bottom elimination*)
10. If we have concluded \( p \), then we can also conclude \( \neg \neg p \) (*double negation introduction*)
11. If we have concluded \( \neg \neg p \), then we can also conclude \( p \) (*double negation elimination*)
Derived Rules

• If we have concluded $\neg q$ and $p \rightarrow q$, we can also conclude $\neg p$ (*modus tollens*)

• Assume $\neg q$ is true. Suppose we assume $p$ and we can then prove $p \rightarrow q$. Then $q$ holds. But this is impossible, so our assumption (that $p$ is true) must be false (*reductio ad absurdum* or *proof by contradiction*)
  • See the implication elimination rule above
Well-Formed Formulas

• A word is a set of symbols using symbols for propositions, connectors, parentheses

• Only some (well-formed formulas or WFFs) are meaningful; these are defined inductively
  • A propositional atom is a WFF
  • Negation of a WFF is a WFF
  • Conjunction of WFFs is a WFF
  • Disjunction of WFFs is a WFF
  • Implication between two WFFs is a WFF
Truth Tables

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<th>( q )</th>
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<th>( p \lor q )</th>
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Equivalence of Formulas: Definitions

• **Sequent** is a set of formulas $\phi_1, \ldots \phi_n$ and a conclusion $\psi$; denoted $\phi_1, \ldots \phi_n \vdash \psi$

• Sequent is *valid* if a proof of it can be found

• $\phi$ and $\psi$ are *provably equivalent* if and only if both $\phi \vdash \psi$ and $\psi \vdash \phi$ hold

• Two formulas are *semantically equivalent* if they have the same truth table values. If $\psi$ evaluates to true whenever $\phi_1, \ldots \phi_n$ evaluate to true, this is denoted $\phi_1, \ldots \phi_n \models \psi$
Soundness and Completeness Theorems

**Soundness Theorem:** Let $\phi_1, \ldots \phi_n$ and $\psi$ be propositional logic formulas. If $\phi_1, \ldots \phi_n \vdash \psi$, then $\phi_1, \ldots \phi_n \models \psi$.

- If, given a set of premises, there is a proof of a conclusion, then the premises and conclusion are semantically equivalent.

**Completeness Theorem:** Let $\phi_1, \ldots \phi_n$ and $\psi$ be propositional logic formulas. If $\phi_1, \ldots \phi_n \models \psi$, then $\phi_1, \ldots \phi_n \vdash \psi$.

- If a set of premises and a conclusion are semantically equivalent, then there is a natural deduction proof for the sequent.
Mathematical Induction

We want to prove a property $M(n)$ holds for all natural numbers $n$
We proceed as follows:

• **BASIS**: prove that $M(1)$ holds

• **INDUCTION HYPOTHESIS**: assert that $M(n)$ holds for $n = 1, \ldots, k$

• **INDUCTION STEP**: prove that if $M(k)$ holds, then $M(k+1)$ holds

Then $M(n)$ is true for all natural numbers $n$. 
Example

• Prove the sum of the first $n$ natural numbers is $\frac{n(n+1)}{2}$.

BASIS: $M(1) = \frac{1(1+1)}{2} = \frac{1(2)}{2} = \frac{2}{2} = 1$, which is clearly true.

INDUCTION HYPOTHESIS: For $n = 1, \ldots, k$, $M(k)$ is true.

INDUCTION STEP: Consider $M(k+1) = 1 + \ldots + k + (k+1)$

$$1 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) \quad \text{induction hypothesis}$$

(continued on next slide)
Example (con’t)

\[ 1 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + (k+1) \]

induction hypothesis

\[ = \frac{k^2}{2} + \frac{k}{2} + \frac{2k}{2} + \frac{2}{2} \]

expanding terms

\[ = \frac{k^2 + 3k + 2}{2} \]

combining terms

\[ = \frac{(k+1)(k+2)}{2} \]

factoring the numerator

\[ = \frac{(k+1)[(k+1)+1]}{2} \]

combining terms

which is \( M(k+1) \), completing the proof
Predicate Logic

• Logic using predicates and quantifiers
• Predicates describe something; quantifiers say what the description applies to

• Quantifiers
  • There exists an \( x \): \( \exists x \)
  • For all \( x \): \( \forall x \)
  • Can combine with \( \neg \) for negation

• Variables
  • *Bound* if quantified with either \( \exists \) or \( \forall \)
  • *Unbound* or *free* if not bound
Examples

• Define:
  • $F(x)$: $x$ is a file
  • $D(y)$: $y$ is a directory
  • $C(x, y)$: directory $y$ contains file $x$

• Then:

$$\forall x \, F(x) \rightarrow (\exists y \, (D(y) \land C(x, y)))$$

says that “every file is contained in a directory”
Formula in Predicate Logic

• If \( p \) is a predicate of \( n \) arguments (\( 1 \leq n \)) and the arguments are terms \( t_1, \ldots, t_n \) defined over the set of functions, then \( p(t_1, \ldots, t_n) \) is a formula

• If \( \phi \) is a formula, then \( \neg \phi \) is also a formula

• If \( \phi \) and \( \varphi \) are formulas, then \( \phi \land \varphi, \phi \lor \varphi, \) and \( \phi \rightarrow \varphi \) are also formulas

• If \( \phi \) is a formula and \( x \) a variable, then \( \forall x \phi \) and \( \exists x \phi \) are also formulas
Rules for Natural Deduction in Predicate Logic

• **Equality**: A term $t$ is equal to itself

• **Substitution**: If $t_1 = t_2$ and $x$ is a free variable in $\phi(x)$, then $f(t_1) = f(t_2)$

• **Universal quantifier elimination**: If you have $\forall x \phi(x)$, then you can replace the $x$ in $\phi(x)$ by any term $t$ that is free in $\phi(x)$

• **Universal quantifier introduction**: If you can prove some formula $\phi(x)$ with $x$ a free variable, then you can derive $\forall x \phi(x)$
Temporal Logic Systems

Introduce notion of time into logic system

• *Linear time logic systems*: events are sequential

• *Branching time logic systems*: events are concurrent ("alternative universes")

Systems view time as:

• *continuous* flow of events

• *discrete* events
Example: Control Tree Logic (CTL)

• Begin with propositional logic

• Add temporal connectives; each uses 2 symbols
  • First symbol: “A”, along all paths; “E”: along at least one path
  • Second symbol: “X”, the next state; “F”, some next state; “G”, all future states; “U”, until some future state

• Precedence rules (high to low)
  • ¬, AG, EG, AF, EF, AX, EX
  • ∧, ∨
  • →
  • AU, EU
Well-Formed Formulas in CTL

- $\top$ (top), $\bot$ (bottom) are formulas
- All atomic descriptions are formulas
- If $\phi$ and $\varphi$ are formulas, then $\phi \land \varphi$, $\phi \lor \varphi$, $\phi \rightarrow \varphi$, $\neg \phi$, $AX\phi$, $EX\phi$, $A[\phi U \varphi]$, $E[\phi U \varphi]$, $AG\phi$, $EG\phi$, $AF\phi$, and $EF\phi$ are also formulas
Semantics of CTL

• A model is $M = (S, \Rightarrow, L)$, where $S$ is a set of states, $\Rightarrow$ is the transition operator on $S$ such that $\forall s \in S \ (\exists s' \in S \ [s \Rightarrow s'])$, $L$ is a labeling function, and $L : S \rightarrow \mathcal{P}(\text{atoms})$
  • $\mathcal{P}(\text{atoms})$ power set of the defined atoms

• Let $M = (S, \Rightarrow, L)$ be a model for CTL. Given any $s \in S$, if a CTL formula $\phi$ holds in state $s$, we write this as $M,s \models \phi$, and say that state $s$ of model $M$ satisfies formula $\phi$.
  • $M,s \not\models \phi$ means state $s$ in model $M$ does not satisfy $\phi$
Rules of CTL

M model, s, s₁, . . . states of M, p atomic proposition of M, φ, φ₁, φ₂ CTL formulas

• ∀s ∈ S [ M, s ⊨ T ]
  • Tautologies hold in all states of M

• ∀s ∈ S [ M, s ⊭ ⊥ ]
  • Tautologies hold in all states of M

• M, s ⊨ p if and only if p ∈ L(s)
  • P holds in state s of M whenever p is in the set of atoms that hold in state s; conversely, if p not in that set, then p does not hold in state s
Rules of CTL

• If $M, s \not\models \phi$, then $M, s \models \neg \phi$
  
  • If a state does not satisfy a formula in the model then it satisfies the negation of the formula

• $M, s \models \phi_1 \land \phi_2$ if and only if $M, s \models \phi_1$ and $M, s \models \phi_2$

• $M, s \models \phi_1 \lor \phi_2$ if and only if $M, s \models \phi_1$ or $M, s \models \phi_2$
  
  • A state in $M$ satisfies the \{and, or\} of two formulas if and only if it satisfies \{both formulas, either formula\} on the right

• $M, s \models \phi_1 \rightarrow \phi_2$ if and only if $M, s \not\models \phi_1$ or $M, s \models \phi_2$
  
  • A state in $M$ satisfies the implication of two formulas if and only if it satisfies the second formula, or neither formula
Rules of CTL

• $M, s \models AX\phi$ if and only if $\forall s_1$ such that $s \rightarrow s_1$ then $M, s_1 \models \phi$

• $M, s \models EX\phi$ if and only if $\exists s_1$ such that $s \rightarrow s_1$ then $M, s_1 \models \phi$
  • A state satisfies a formula in some next state if and only if {every, at least one} state implied by the original state also satisfies the formula

• $M, s \models AG\phi$ if and only if, for all paths $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \ldots$, where $s = s_1$ and $\forall s_i$ on the path, $[M, s_i \models \phi]$
  • A state satisfies a formula in some next state if and only if every state implied by the original state also satisfies the formula

• $M, s \models EG\phi$ if and only if there exists a path $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \ldots$, where $s = s_1$ and $\forall s_i$ on the path, $[M, s_i \models \phi]$
  • A path with all states satisfying a formula exists if and only if every state on the path beginning at the original state satisfies the formula
Rules of CTL

• $M, s \models AF\phi$ if and only if for all paths $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \ldots$, where $s = s_1$ and $\exists s_i [M, s_i \models \phi]$
  • On all paths, there will be a state satisfying the formula if and only if every path of transitions beginning at the original state contains at least one state that satisfies the formula

• $M, s \models EF\phi$ if and only for all paths $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \ldots$, where $s = s_1$ and $\exists s_i$ on the path $[M, s_i \models \phi]$
  • There is a path with one state satisfying the formula if and only if a state on a path of transitions beginning at the original state satisfies the formula
Rules of CTL

• $M, s \models A[\phi U \phi]$ if and only if for all paths $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \ldots,$
  \[ \exists i \ [i \geq 0 \land s_i \models \phi_2 \text{ and } \forall j \ [0 \leq j < i \rightarrow s_j \models \phi_1] \]  
  • On all paths, there will be a state satisfying the formula if and only if every path of transitions beginning at the original state has a state satisfying the second formula and all previous states in that path satisfy the first formula

• $M, s \models E[\phi U \phi]$ if and only if for some path $s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow \ldots,$
  \[ \exists i \ [i \geq 0 \land s_i \models \phi_2 \text{ and } \forall j \ [0 \leq j < i \rightarrow s_j \models \phi_1] \]  
  • There is a path on which there is a state satisfying the formula if and only if every path of transitions beginning at the original state has a state satisfying the second formula and all previous states in that path satisfy the first formula