ECS 235B Module 17
Lattices

Overview

- Lattices used to analyze several models
    - Bell-LaPadula confidentiality model
    - Biba integrity model
- A lattice consists of a set and a relation
- Relation must partially order set
    - Relation orders some, but not all, elements of set

Sets and Relations

- S set, R: $S \times S$ relation
  - If a, $b \in S$, and $(a, b) \in R$, write aRb
- Example
  - $I = \{ 1, 2, 3 \}$; R is $\leq$
  - $R = \{ (1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3) \}$
  - So we write $1 \leq 2$ and $3 \leq 3$ but not $3 \leq 2$

Relation Properties

- Reflexive
  - For all $a \in S$, aRa
  - On I, ≤ is reflexive as $1 \le 1$, $2 \le 2$, $3 \le 3$
- Antisymmetric
  - For all $a, b \in S$, $aRb \wedge bRa \Rightarrow a = b$
  - On I, ≤ is antisymmetric as $1 \le x$ and $x \le 1$ means $x = 1$
- Transitive
  - For all $a, b, c \in S$, $aRb \wedge bRc \Rightarrow aRc$
  - On I, ≤ is transitive as $1 \le 2$ and $2 \le 3$ means $1 \le 3$

Example

- $\mathbb{C}$ set of complex numbers
- $a \in \mathbb{C} \Rightarrow a = a_{\mathbf{R}} + a_{\mathbf{I}}i$, where $a_{\mathbf{R}}$, $a_{\mathbf{I}}$ integers
- $a \leq_{\mathbf{C}} b$ if, and only if, $a_{\mathbf{R}} \leq b_{\mathbf{R}}$ and $a_{\mathbf{I}} \leq b_{\mathbf{I}}$
- $a \leq_{\mathbf{C}} b$ is reflexive, antisymmetric, transitive
  - As $\leq$ is over integers, and $a_{\mathbf{R}}$, $a_{\mathbf{I}}$ are integers

Partial Ordering

- Relation *R* orders some members of set *S*
  - If all ordered, it's a total ordering
- Example
  - $\leq$ on integers is total ordering
  - $\leq_{\mathbb{C}}$ is partial ordering on $\mathbb{C}$
    - Neither $3+5i \leq_{\mathbb{C}} 4+2i$ nor $4+2i \leq_{\mathbb{C}} 3+5i$ holds

Upper Bounds

- For a, $b \in S$, if u in S with aRu, bRu exists, then u is an *upper bound*
  - A *least upper bound* if there is no $t \in S$ such that aRt, bRt, and tRu
- Example
  - For $1 + 5i$, $2 + 4i \in \mathbb{C}$
    - Some upper bounds are $2 + 5i$, $3 + 8i$, and $9 + 100i$
    - Least upper bound is $2 + 5i$

Lower Bounds

- For a, $b \in S$, if l in S with lRa, lRb exists, then l is a *lower bound*
  - A *greatest lower bound* if there is no $t \in S$ such that tRa, tRb, and lRt
- Example
  - For $1 + 5i$, $2 + 4i \in \mathbb{C}$
    - Some lower bounds are 0, $-1 + 2i$, $1 + 1i$, and $1+4i$
    - Greatest lower bound is $1 + 4i$

Lattices

- Set *S*, relation *R*
  - *R* is reflexive, antisymmetric, transitive on elements of *S*
  - For every $s, t \in S$, there exists a greatest lower bound under *R*
  - For every $s, t \in S$, there exists a least upper bound under *R*

Example

- $S = \{ 0, 1, 2 \}$; $R = \leq$ is a lattice
  - R is clearly reflexive, antisymmetric, transitive on elements of S
  - Least upper bound of any two elements of S is the greater of the elements
  - Greatest lower bound of any two elements of S is the lesser of the elements

Picture

$$\begin{tikzcd} 2 \arrow[d] \\ 1 \arrow[d] \\ 0 \end{tikzcd}$$

Arrows represent ≤; this forms a total ordering

Example

- $\mathbb{C}$, $\leq_{\mathbb{C}}$ form a lattice
  - $\leq_{\mathbb{C}}$ is reflexive, antisymmetric, and transitive
    - Shown earlier
  - Least upper bound for *a* and *b*:
    - $c_{\mathbf{R}} = \max(a_{\mathbf{R}}, b_{\mathbf{R}})$, $c_{\mathbf{I}} = \max(a_{\mathbf{I}}, b_{\mathbf{I}})$; then $c = c_{\mathbf{R}} + c_{\mathbf{I}} i$
  - Greatest lower bound for *a* and *b*:
    - $c_{\mathbf{R}} = \min(a_{\mathbf{R}}, b_{\mathbf{R}})$, $c_{\mathbf{I}} = \min(a_{\mathbf{I}}, b_{\mathbf{I}})$; then $c = c_{\mathbf{R}} + c_{\mathbf{I}} i$

Picture

$$
\begin{tikzcd}
 & 2+5i \arrow[dl] \arrow[dr] \arrow[dd] & \\
1+5i \arrow[dr] & & 2+4i \arrow[dl] \\
 & 1+4i &
\end{tikzcd}
$$

Arrows represent $\leq_{\mathbb{C}}$